Geometrical optics of Brownian motion

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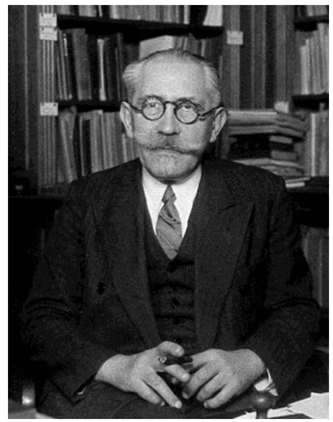
Outline

- Large deviations of Brownian motion can be described by geometrical optics
- Example 1: winding angle around a disk (presented yesterday by Naftali Smith)
- Example 2: survival of Brownian motion against an invading wall
- Example 3: Airy distribution and additional statistics
- Summary

Brownian motion came to prominence in physics in 1905-1908

A. Einstein



P. Langevin

J. Perrin

Today, more than a hundred years since those remarkable discoveries, Brownian motion is a central paradigm in physics, chemistry, biology, computer science, finance, *etc*.

Path-integral formulation for the Wiener process

Probability of a Brownian path x(t)

$$\text{Prob}[x(t)] \sim \exp\left[-\frac{1}{4D}\int_0^T \left(\frac{dx}{dt}\right)^2 dt\right] \qquad (1)$$

When this probability is exponentially small, it is often dominated by a single Brownian path x(t) – the optimal path, for which the action functional

$$S = \frac{1}{4D}\int_0^T \left(\frac{dx}{dt}\right)^2 dt$$

is minimal subject to problem-specific constraints

Geometrical optics of Brownian motion = saddle-point evaluation of the path integral (1)

Formally, $D \to 0$. Similar to the limit $\hbar \to 0$ in WKB

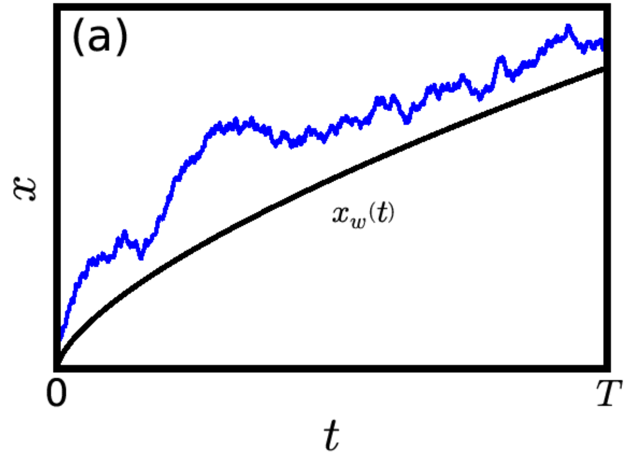
Example 2: survival of Brownian motion against invading wall

BM and N. R. Smith, J. Phys. A: Math. Theor. **52**, 415001 (2019)

A Brownian particle is released at $t=0$ at $x=\varepsilon>0$. An absorbing wall, initially at $x=0$, is moving to the right according to a power law

$$x_w(t) = Ct^\gamma, \qquad \gamma > 0, C > 0$$

What is the probability P(T) that, at long time T, the particle has not yet been absorbed?



Early work by mathematicians: A.A. Novikov, Math. USSR Sb. **38**, 495 (1981) and references there, and by physicists: P.L. Krapivsky and S. Redner (1996,1999)

The survival probability strongly depends on γ

$\gamma < \frac{1}{2}$: P(T) goes down as $T^{-1/2}$, that is only as a power law. This regime is beyond geometrical optics

$\gamma > \frac{1}{2}$: P(T) is exponentially small, and it can be described by geometrical optics:

Minimize $S = \frac{1}{4D}\int_0^T \left(\frac{dx}{dt}\right)^2 dt$

under conditions $x(0) = L,\ x(t) > x_w(t) = C\,t^\gamma$

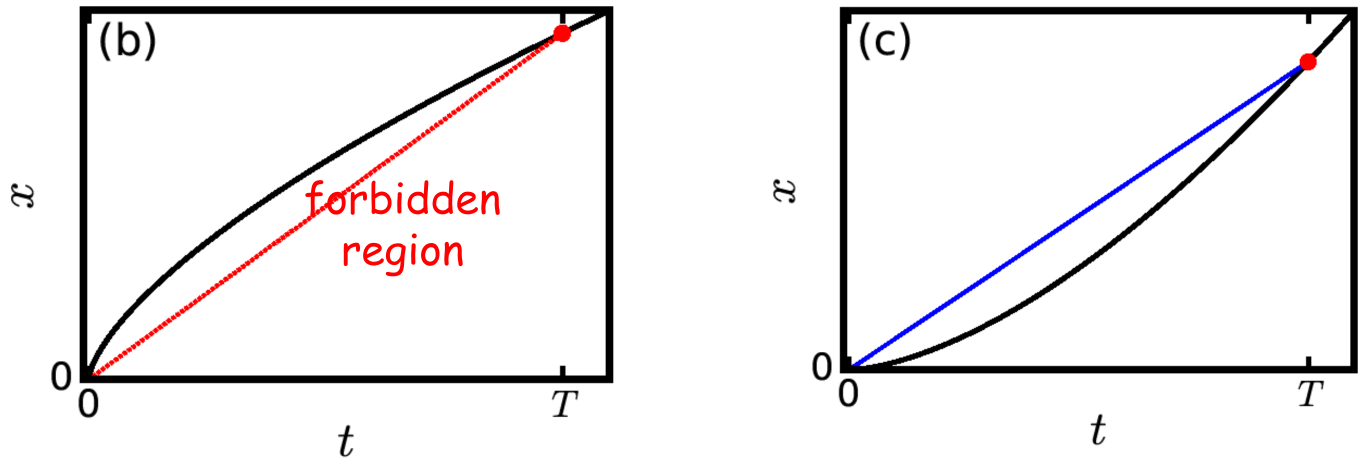
One-sided variations

Minimize $$S = \frac{1}{4D}\int_0^T \left(\frac{dx}{dt}\right)^2 dt$$

under conditions $x(0) = L, x(t) > x_w(t) = C\, t^\gamma$

One-sided variations

$1/2 < \gamma < 1$ $\qquad\qquad$ $\gamma > 1$



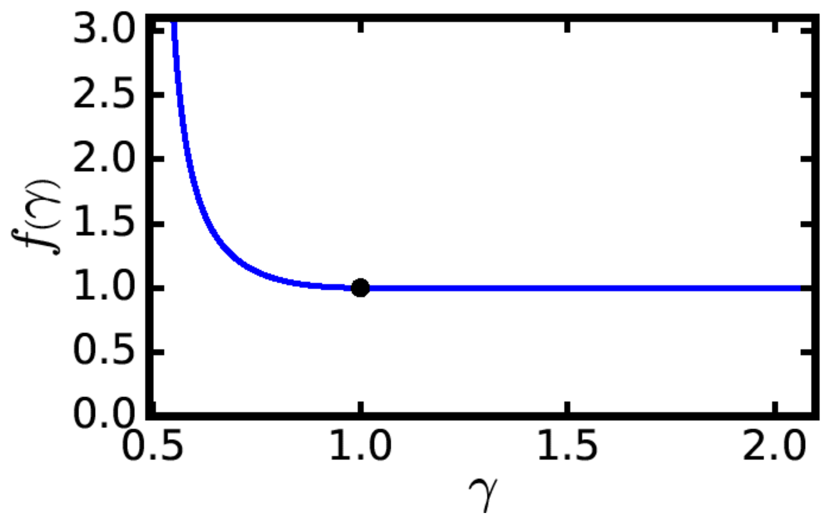
Optimal path: $x(t) = x_w(t) = C\, t^\gamma$ $\qquad$ Optimal path: $x(t) = C\, T^{\gamma-1} t$

Evaluate the action $S = \frac{1}{4D}\int_0^T \left(\frac{dx}{dt}\right)^2 dt$

$$-\ln P(T) \cong S = \frac{C^2 f(\gamma) T^{2\gamma-1}}{4D}$$

$$f(\gamma) = \begin{cases} \frac{\gamma^2}{2\gamma - 1}, & 1/2 < \gamma \le 1 \\ 1, & \gamma \ge 1 \end{cases}$$

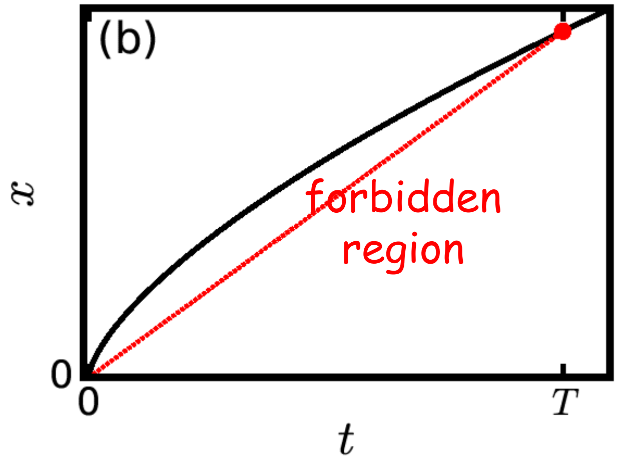
Agrees with exact result for $\gamma = 1$



Applicability: large action, hence long-time limit $T \to \infty$

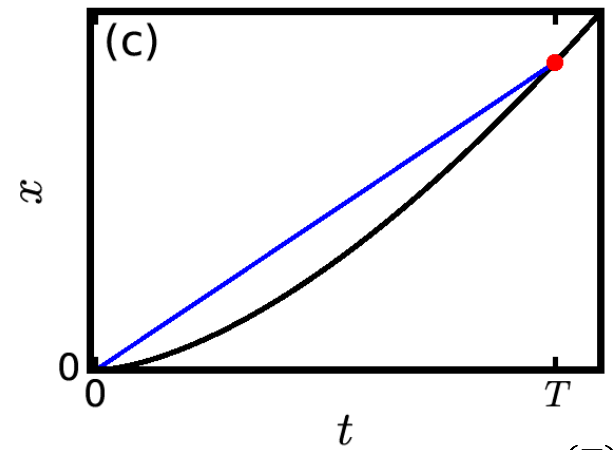
The solution is purely geometrical and not limited to power-law walls

$x_w(t)$ convex upward



Optimal path: $x(t) = x_w(t)$

$x_w(t)$ convex downward

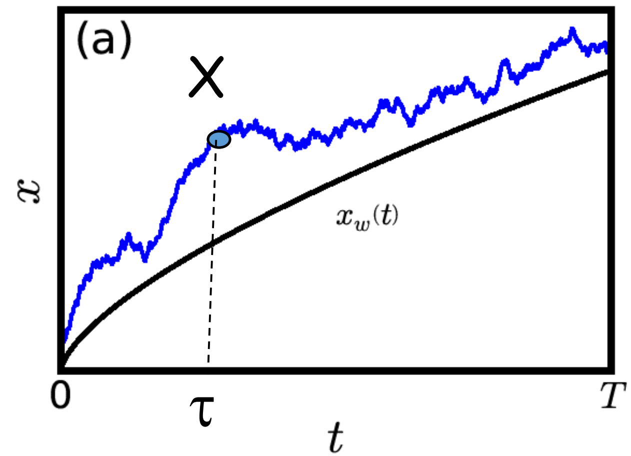


Optimal path: $x(t) = \frac{x_w(T)}{T} t$

coincides with Novikov (1981), but here it is a one-line calculation

$$-\ln P(T) \cong S = \begin{cases} \frac{1}{4D} \int_0^T \left(\frac{dx_w}{dt} \right)^2 dt\,, & x_w(t) \text{ convex upward} \\ \frac{x_w(T)^2}{4DT}, & x_w(t) \text{ convex downward} \end{cases}$$

A more subtle question: What is the position distribution $P(X, \tau, T)$ at intermediate time τ, given that the particle is not absorbed until time T?

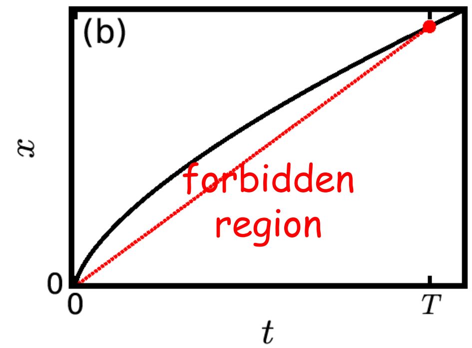


Minimize $S = \frac{1}{4D} \int_0^T \left(\frac{dx}{dt} \right)^2 dt$

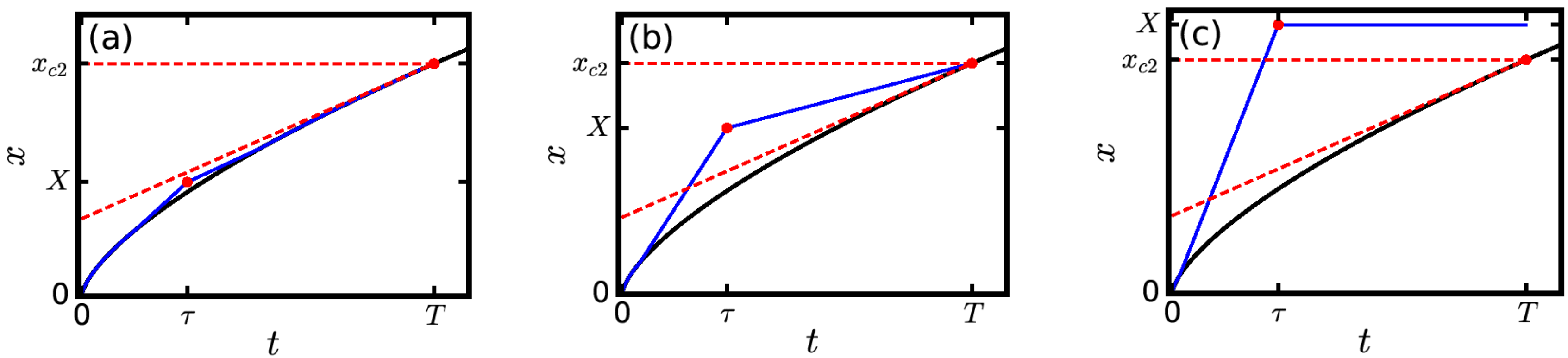
under conditions $x(0) = L, x(t) > x_w(t) = C\, t^\gamma$ and $x(\tau) = X$,

In geometrical optics, this conditional probability is equal to the ratio of the probabilities (and the action is equal to the difference of the actions) of two different optimal paths that avoid absorption: with and without the constraint $x(\tau) = X$

The regime $\frac{1}{2} < \gamma < 1$ is especially interesting



(a) and (b): tangent constructions for one-sided variations



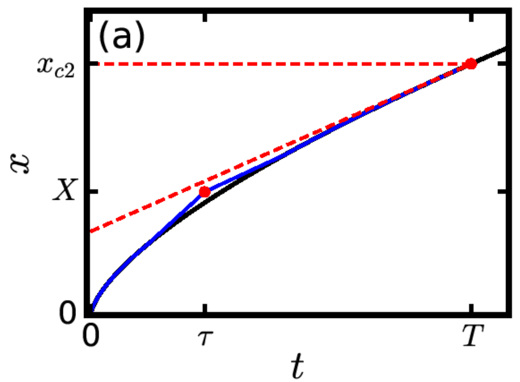
Three distinct regions of X: subcritical, first supercritical and second supercritical

$$-\ln P(X,\tau,T) \cong \frac{x_w(T)^2}{DT}\varphi\left(\frac{X}{x_w(T)},\frac{\tau}{T}\right)$$

φ is a non-analytic function of X at $X=X_{c1}$ and $X=X_{c2}$. Such non-analyticities are called dynamical phase transitions. Here they are of second and third order, respectively.

Sharp transitions appear only in the limit $T \to \infty$

Let's look more closely at the subcritical regime



$$\frac{1}{2} < \gamma < 1,$$
$$T \to \infty$$

What happens when $x(\tau) = X$ is close to $x_w(\tau) = C\tau^\gamma$?
Asymptote of geometrical-optics result:

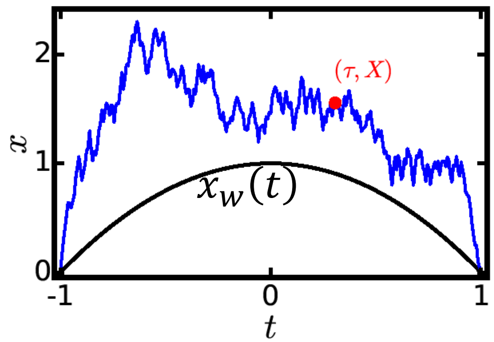
$$-\ln P(X,\tau) \cong \frac{2\sqrt{2\,\gamma(1-\gamma)}C^2\tau^{2\gamma-1}}{3D}\left(\frac{X}{C\tau^\gamma} - 1\right)^{3/2} \qquad (1)$$

No dependence on T: the conditional distribution is local in time!

Equation (1) coincides (up to a pre-exponent) with the tail of the Ferrari-Spohn distribution

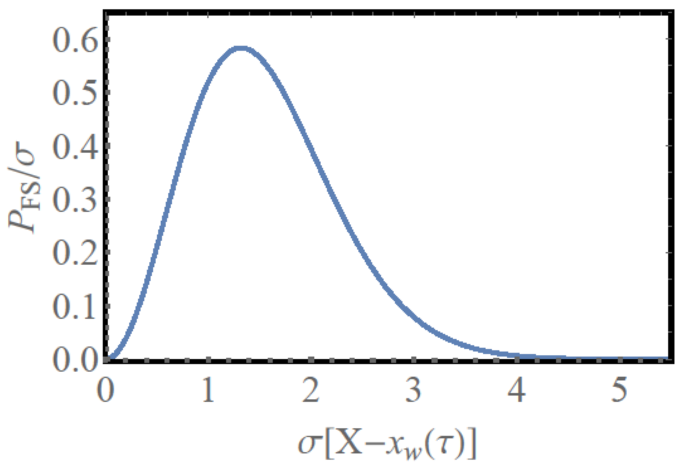
The Ferrari-Spohn distribution

P.L. Ferrari and H. Spohn, Ann. Probab. **33**, 1302 (2005)



A Brownian excursion $x(t)$, conditioned to stay away from a swinging wall $x_w(t)$

At $T \to \infty$, typical fluctuations of $\Delta X = X - x_w(\tau)$ are distributed according to a distribution that depends only on the second derivative $\ddot{x}_w(\tau)$.



$$P_{FS}(\Delta X, \tau) = \frac{\sigma \mathrm{Ai}[\sigma\, \Delta X + a_1]^2}{\mathrm{Ai}'(a_1)^2},$$

$$\sigma = \left[\frac{-\ddot{x}_w(\tau)}{2D^2}\right]^{1/3}$$

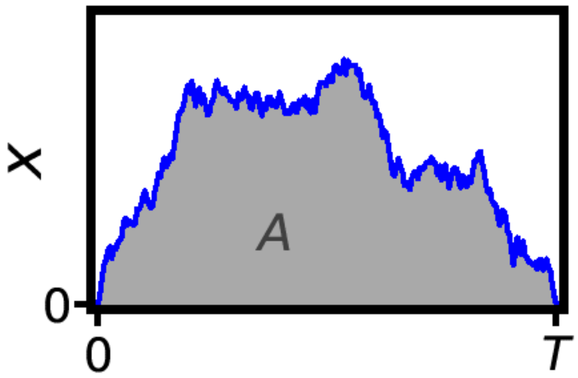
Ai(...) is the Airy function,
a_1 =-2.33810... is its first root

The Ferrari-Spohn (FS) distribution $P_{FS}(\Delta X, \tau)$ and our large-deviation tail $P(X, \tau)$ have a joint validity region. This is a large-ΔX asymptote of the FS dist., but a small-ΔX asymptote of geometrical optics

Example 3: a tail of the Airy distribution

T. Agranov, P. Zilber, N.R. Smith, T. Admon, Y. Roichman and BM, Phys. Rev. Res. (in press), arXiv:1908.08354

The Airy distribution is the distribution of the area $A_T = \int_0^T x(t)dt$ under a Brownian excursion



D. A. Darling, Ann. Probab. **11**, 803 (1983).
G. Louchard, J. Appl. Probab. **21**, 479 (1984).

Many applications in computer science. More recently, in physics:

Height of fluctuating interfaces S. N. Majumdar and A. Comtet, Phys. Rev. Lett. **92**, 225501 (2004); J. Stat. Phys. **119**, 314 (2005).
Sizes of avalanches in sand pile models M. A. Stapleton and K. Christensen J. Phys. A: Math. Gen. **39**, 9107 (2006).
Sizes of ring polymers S. Medalion, E. Aghion, H. Meirovitch, E. Barkai and D. A. Kessler, Sci. Rep. **6**, 27661 (2016).
Positions of laser-cooled atoms E. Barkai, E. Aghion, and D. A. Kessler, Phys. Rev. X **4**, 021036 (2014).

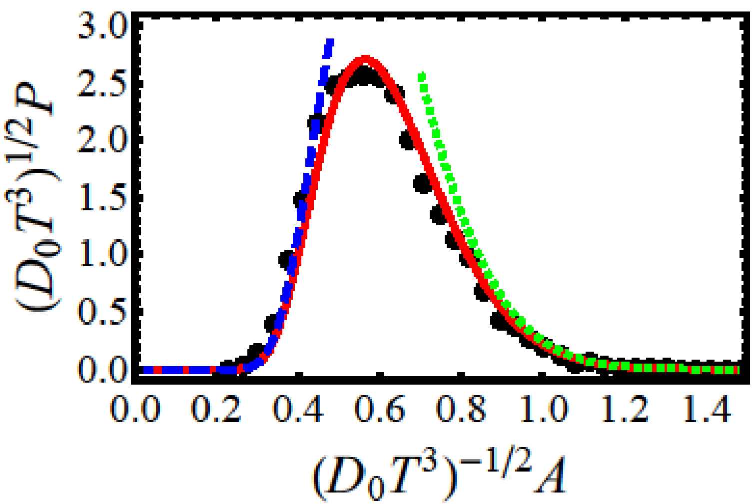
The Airy distribution

$$P(A,T) = \frac{1}{\sqrt{D\,T^3}} f\left(\frac{A}{\sqrt{D\,T^3}}\right)$$

$$f(\xi) = \frac{2\sqrt{6}}{\xi^{10/3}} \sum_{k=1}^{\infty} e^{-\beta_k/\xi^2} \beta_k^{\,2/3} U\left(-\frac{5}{6}, \frac{4}{3}, \frac{\beta_k}{\xi^2}\right),$$

$\beta_k = \frac{2\,\alpha_k^3}{27}$. α_k are ordered abs. values of zeros of the Airy function Ai(ξ).

L. Takács, Adv. Appl. Prob. **23**, 557 (1991); J. Appl. Prob. **32**, 375 (1995).



• experimental data

$$-\ln P(A,T) \cong \begin{cases} \frac{2\,\alpha_1{}^3}{27}\frac{DT^3}{A^2}, & A \ll \sqrt{D\,T^3} \\ \frac{6\,A^2}{DT^3}, & A \gg \sqrt{D\,T^3} \end{cases}$$

The $A \gg \sqrt{D\,T^3}$ tail can be obtained from geometrical optics

Minimize $$S = \frac{1}{4D}\int_0^T \left(\frac{dx}{dt}\right)^2 dt$$

under conditions $x(0) = x(T) = 0$, $x(0 < t < T) > 0$, $\int_0^T x(t)dt = A$

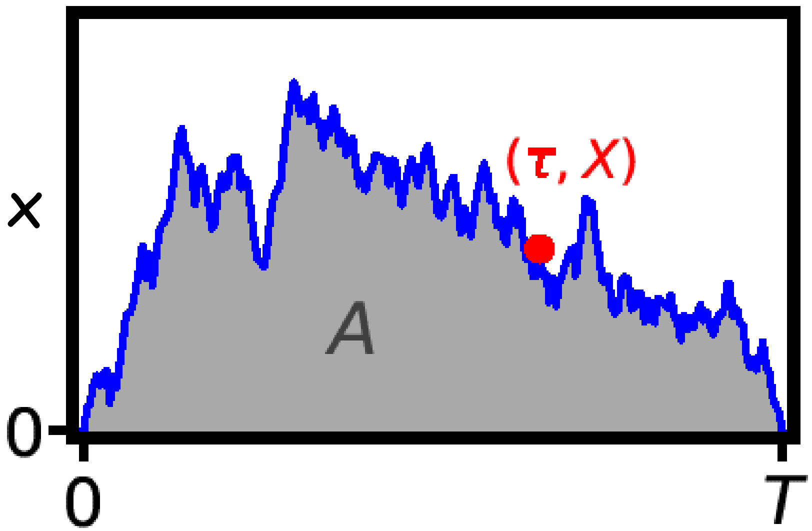
The constrained action $$S = \int_0^T \left[\frac{1}{4D}\left(\frac{dx}{dt}\right)^2 - \lambda x\right] dt$$ λ: Lagrange multiplier

Optimal path is a parabola: $x(t) = \left(\frac{6At}{T^2}\right)\left(1 - \frac{t}{T}\right)$

One-sidedness plays no role

$$S = \frac{6\,A^2}{DT^3}$$ correctly reproduces the $A \gg \sqrt{D\,T^3}$ tail

A more subtle question: What is the position distribution $P(X,\tau,T,A)$ of the Brownian excursion at intermediate time τ conditioned on area A?



From dimensional analysis

$$P(X,\tau,T,A) = \frac{T}{A} F\left(\frac{TX}{A}, \frac{t}{T}, \frac{A}{\sqrt{D\,T^3}}\right)$$

$$\frac{A}{\sqrt{D\,T^3}} \gg 1$$

Minimize $$S = \frac{1}{4D}\int_0^T \left(\frac{dx}{dt}\right)^2 dt$$

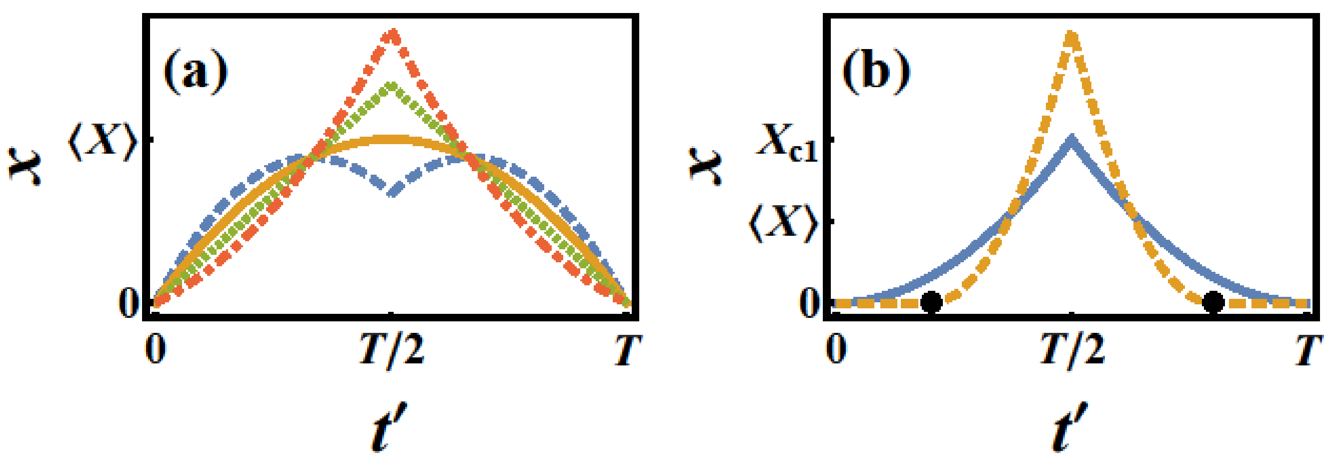
under conditions $x(0) = x(T) = 0$, $x(0 < t < T) > 0$, $\int_0^T x(t)dt = A$, $x(\tau) = X$

In geometrical optics, this conditional probability is equal to the ratio of the probabilities (and the action is equal to the difference of the actions) of two different optimal paths that enclose area A: with and without the constraint $x(\tau) = X$

Minimize $\quad S = \frac{1}{4D}\int_0^T \left(\frac{dx}{dt}\right)^2 dt$

under conditions $x(0) = x(T) = 0$, $x(0 < t < T) > 0$, $\int_0^T x(t)dt = A$, $x(T/2) = X$

The optimal path is composed of two parabolic segments

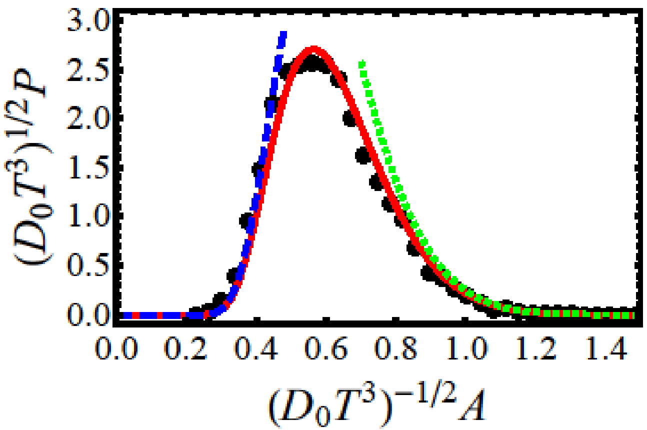


At $X > X_c = \frac{3A}{T}$ the one-sidedness kicks in via tangent construction. This leads to a dynamical phase transition of third order

For $\tau \neq T/2$ there are *two* third-order transitions, see Agranov *et al.* arXiv:1908.08354.
Sharp transitions appear only in the limit $\frac{A}{\sqrt{D\,T^3}} \to \infty$; they are smoothed out at finite $\frac{A}{\sqrt{D\,T^3}}$

Does geometrical optics describe all large deviations of Brownian motion?
The answer is no.

Example: the other tail of the Airy distribution



$$-\ln P(A,T) \cong \begin{cases} \frac{2\,{\alpha_1}^3}{27}\,\frac{DT^3}{A^2}, & A \ll \sqrt{D\,T^3} \\ \frac{6\,A^2}{DT^3}, & A \gg \sqrt{D\,T^3} \end{cases}$$

The $A \ll \sqrt{D\,T^3}$ tail follows from a different large-deviation formalism

Let $\bar{x} = \frac{A_T}{T} = \frac{1}{T}\int_0^T x(t)dt \qquad C\frac{DT^3}{A^2} = Tf(a),\ a = \frac{A}{T}, f(a) = C\,\frac{D}{a^2},\ \ \bar{x} \ll \sqrt{DT}$

Donsker-Varadhan large deviation principle.

The constant $C = \frac{2\,{\alpha_1}^3}{27}$ can be found with the tilted generator technique, see e.g. H. Touchette, Physica A **504**, 5 (2018)

Here there are many untypical paths which lead to small $\bar{x}$

Other recent works on geometrical optics of Brownian motion

1. BM, Large fluctuations of the area under a constrained Brownian excursion, J. Stat. Mech. (2019) 013210.

2. N. R. Smith and BM, Geometrical optics of constrained Brownian excursion: from the KPZ scaling to dynamical phase transitions, J. Stat. Mech. (2019) 023205.

3. BM, Mortal Brownian motion: Three short stories, Int. J. Mod. Phys. B **33**, 1950172 (2019).

4. S. N. Majumdar and BM, Statistics of first-passage Brownian functionals, J. Stat. Mech. (in press); arXiv:1911.06668.

Relatives of geometrical optics of Brownian motion:

optimal fluctuation method, weak-noise theory, macroscopic fluctuation theory, …

These are classical field theories

Summary

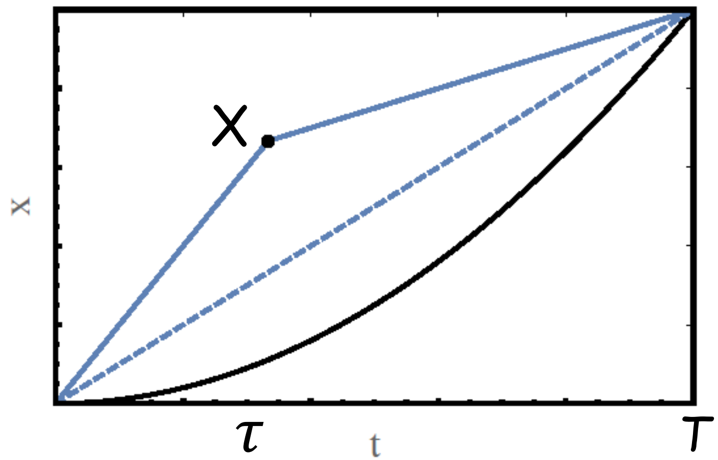
- Geometrical optics is a simple and efficient tool for studying Brownian motion, “pushed” to a large-deviation regime by constraints.

- New dynamical phase transitions, of purely geometrical origin.

- The Ferrari-Spohn distribution appears in a whole class of settings where conditioned optimal paths of Brownian motion are localized at smooth obstacles.

- Future work: applications to stochastic search problems, analogs of geometrical optics in non-Markov processes, more extensions to stochastic fields, ….

$$\gamma > 1, \qquad T \to \infty$$



$$-\ln P(X,\tau,T) \cong \frac{x_w(T)^2}{4DT} \frac{\left(\frac{X}{x_w(T)} - \frac{\tau}{T}\right)^2}{\frac{\tau}{T}\left(1-\frac{\tau}{T}\right)}$$

A Gaussian distribution with maximum at $X = (\tau/T)\, x_w(T)$, that is on the unconditioned path

In this form the result is valid for *any* wall function which is convex downward:

$$\ddot{x}_w(t) > 0$$

Mathematical definition of Brownian motion x(t) is given in terms of Wiener process

$$\frac{dx}{dt} = \xi(t)$$

$\xi(t)$ Gaussian white noise

$$< \xi(t) > \ = 0, \quad < \xi(t_1)\xi(t_2) > = 2\mathrm{D}\,\delta(\mathrm{t}_1 - \mathrm{t}_2)$$

D diffusion constant

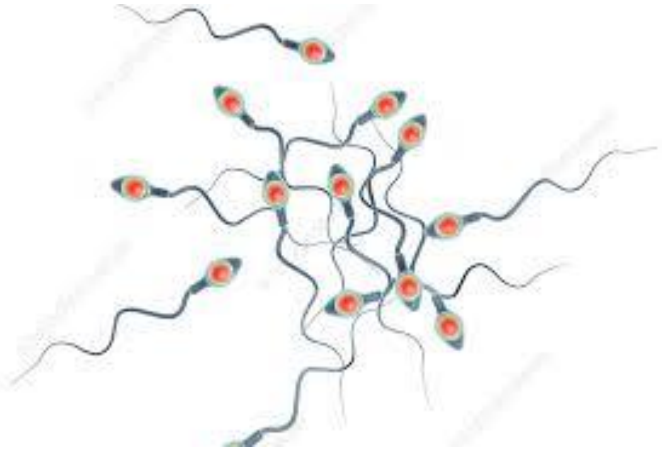
The probability distribution P(x,t) obeys the diffusion equation

$$\frac{\partial P(x,t)}{\partial t} = D\,\frac{\partial^2 P(x,t)}{\partial x^2}$$

Short-time large deviations are intrinsic in the problem of $N \gg 1$ "searchers"

Redundancy in biology: a huge copy number of agents (molecules, ions, …) is often needed in situations where only one agent does the job.

A striking example: 3×10^8 sperm cells initially attempt to reach the oocyte after copulation in humans, and only one (rarely, a few) make it.



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Why so a huge redundancy? One possible reason is to reduce the random search time

B. Meerson and S. Redner, Phys. Rev. Lett. **114**, 198101 (2015); Z. Schuss, K. Basnayake and D. Holcman, Phys. Life Rev. **28**, 52 (2019).

The first arrival is unusually fast: a large deviation, a simple optimal path